# MATH Math 162A Review: Vector Calculus

1. Let $A$ be a square matrix, and let

$$\sin A = A - \frac{A^3}{3!} + \frac{A^5}{5!} - \frac{A^7}{7!} + \cdots$$

Prove that

$$\frac{d\sin(At)}{dt^2} + A^2\sin(At) = 0$$

**Solution:** We can use the same method as in the video to prove the result directly, or we can use the following method.

$$e^{iAt} = \cos At + i\sin At$$

Since

$$\frac{d}{dt} e^{iAt} = iA e^{iAt}$$

$$\frac{d^2}{dt^2} e^{iAt} = (iA)^2 e^{iAt} = -A^2 e^{iAt}$$

Thus

$$\frac{d^2}{dt^2}(\cos At + i\sin At) = -A^2(\cos At + i\sin At)$$

Take the imaginary part, we get

$$\frac{d^2}{dt^2}\sin At = -A^2 \sin At$$

2. Let $V = (v_1, v_2, v_3)$ be an $\mathbb{R}^3$-valued function of $x, y, z$. The $\operatorname{curl} V$ is defined by

$$\operatorname{curl} V = \left( \left( \frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z} \right), \left( \frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x} \right), \left( \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \right) \right)$$

The divergence $\operatorname{div} V$ is defined by

$$\operatorname{div} V = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

Prove that

$$\operatorname{div} \operatorname{curl} V = 0.$$

**Solution:**

$$\operatorname{div} \operatorname{curl} V = \operatorname{div} \left( \frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}, \frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}, \frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y} \right)$$

$$= \frac{\partial}{\partial x} \left( \frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z} \right) + \frac{\partial}{\partial y} \left( \frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x} \right) + \frac{\partial}{\partial z} \left( \frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y} \right)$$

$$= \cancel{\frac{\partial^2 V_3}{\partial x \partial y}} - \cancel{\frac{\partial^2 V_2}{\partial x \partial z}} + \frac{\partial^2 V_1}{\partial y \partial z} - \cancel{\frac{\partial^2 V_3}{\partial y \partial x}}$$

$$+ \cancel{\frac{\partial^2 V_2}{\partial z \partial x}} - \frac{\partial^2 V_1}{\partial z \partial y} = 0$$